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LETTER TO THE EDITOR

Duality properties of a general vertex model†

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Abstract. We consider the duality properties of a general vertex model on a lattice in any spatial dimension. The analysis is based on a generalised weak-graph transformation under which the partition function of the vertex model remains invariant. It is shown that the generalised weak-graph transformation is self-dual for lattice coordination number $q = 2$, 3, 4, 5, 6, and we conjecture that the self-dual property holds for general q. We also obtain the self-dual manifold for $q = 3$, 4, and it is found that, in an Ising subspace, the manifold coincides with the known Ising critical locus.

Consider a vertex model on a lattice $\mathscr{L}$, which can be in any spatial dimension, of E edges and with coordination number (valency) q. A line graph on $\mathscr{L}$ is a collection of a subset of the edges, which, if regarded as being covered by bonds, generates bond configurations at all vertices. With each vertex we associate a weight according to the configuration of the incident bonds. This gives rise to a 2^q-vertex model whose partition function is

$$Z = \sum_G \prod_{i=1}^{N} \omega_i \tag{1}$$

where ω_i is the weight of the ith vertex. The summation is taken over all 2^E line graphs G on $\mathscr{L}$.

The expression (1) defines a very general vertex model which encompasses many outstanding lattice statistical problems. For example, the Ising model in a non-zero magnetic field formulated in the usual high-temperature (tanh) expansion is a 2^q-vertex problem (see, e.g., Lieb and Wu 1972). It can also be shown that the eight-vertex model for $q = 3$ (Wu 1974b, Wu and Wu 1988a) as well as another special case of the general q problem (Wu 1972, 1974a) are completely equivalent to an Ising model in a non-zero magnetic field, a property that has been used to deduce the critical locus for the vertex models in question (Wu 1974a, b). However, very little is known about other properties of these vertex models.

In this letter we report some new results on duality properties for this 2^q-vertex model. We show that a generalised weak-graph transformation, which leaves the partition function unchanged, is always self-dual, and obtain the self-dual manifold (locus) for $q = 3$, 4. We further show that this self-dual locus coincides with the critical locus in the ferromagnetic Ising subspace.

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For simplicity, we consider a *symmetric* version of the model for which the vertex weight depends only on the number of bonds incident to the vertex. It should be noted this is not a severe restriction, since the analysis can be extended in a straightforward fashion to the general (asymmetric) case at the expense of a generalised weak-graph transformation of the vertex weights under which the partition function remains invariant. The weak-graph expansion was first used by Nagle (1968) in an analysis of the series expansion of six-vertex models. A general formulation of the weak-graph expansions given by Wegner (1973) permits the introduction of a free parameter into the formulation, a fact first recognised and explicitly used in the analysis of the eight-vertex model (Wu 1974b). To emphasise the extra degree of freedom introduced by the free parameter, we shall refer to the transformation containing free parameter(s) as the *generalised* weak-graph transformation†.

Consider first the case of $q=3$, namely an eight-vertex model whose vertex configurations and weights are shown in figure 1. The symmetric eight-vertex model has been considered previously (Wu 1974b, Wu and Wu 1988a), and it was established that, for a, b, c, d real, the vertex problem is completely equivalent to a ferromagnetic Ising model in a real magnetic field or an antiferromagnetic Ising model in a pure imaginary field. Using this Ising equivalence, the critical manifold of the eight-vertex model in the ferromagnetic Ising subspace is found to be‡

$$f(a, b, c, d)=0 \tag{2}$$

where

$$f(a, b, c, d) \equiv a(b^3+d^3)-d(a^3+c^3)+3(ab+bc+cd)(c^2-bd+ac-b^2). \tag{3}$$

We now show that the critical manifold (2) can also be obtained directly from an analysis of the self-dual property of the eight-vertex model.

The generalised weak-graph transformation for $q=3$ is (Wu 1974b)

$$\tilde{\boldsymbol{A}}=\boldsymbol{W}\boldsymbol{A} \tag{4}$$

$$\tilde{\boldsymbol{A}}=\begin{pmatrix}\tilde{a}\\ \tilde{b}\\ \tilde{c}\\ \tilde{d}\end{pmatrix} \qquad \boldsymbol{A}=\begin{pmatrix}a\\ b\\ c\\ d\end{pmatrix}$$
$$\boldsymbol{W}=\frac{1}{(1+y^2)^{3/2}}\begin{pmatrix}1 & 3y & 3y^2 & y^3\\ y & 2y^2-1 & y^3-2y & -y^2\\ y^2 & y^3-2y & 1-2y^2 & y\\ y^3 & -3y^2 & 3y & -1\end{pmatrix} \tag{5}$$

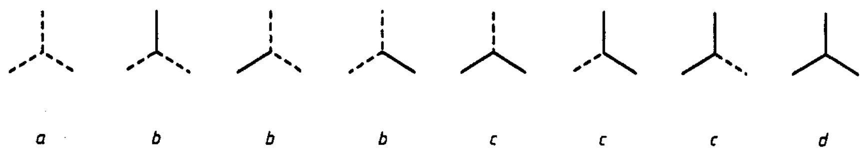

Figure 1. Vertex configurations and weights for the symmetric eight-vertex model.

† In general, more than one parameter is needed in the analysis of the asymmetric model.
‡ See, in particular, footnote 5 of Wu and Wu (1988a).

where y is arbitrary. The partition function (1) is invariant under the transformation (4); namely we have

$$Z(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = Z(a, b, c, d). \tag{6}$$

Generally, a transformation is self-dual if it possesses a fixed point, i.e. if it maps a point in the parameter space $\{a, b, c, d\}$ onto itself. For a transformation whose coefficients contain a parameter such as y in (4), we generally expect the transformation to be self-dual only for some special values of y. However, we now show that the generalised weak-graph transformation (4) is *always* self-dual, i.e. there exist fixed points for *all* y! We further determine the manifold in the parameter space containing all such fixed (self-dual) points.

Consider first the more general eigenvalue equation

$$\boldsymbol{W}\boldsymbol{A} = \lambda \boldsymbol{A} \tag{7}$$

where λ is the eigenvalue of $\boldsymbol{W}$. Combining (4) with (7), we see that the transformation $\boldsymbol{W}$ is self-dual if $\lambda = 1$. However, the transformation for $\lambda = -1$ can also be regarded as 'self-dual', since in this case the net effect of (7) is to negate all vertex weights. This introduces a factor $(-1)^N$ into the overall Boltzmann factor, and does not change anything as we generally have $N = $ even.

The characteristic equation of (7) is

$$\det|W_{ij} - \lambda\delta_{ij}| = 0 \tag{8}$$

where $i, j = 1, 2, 3, 4$, and W_{ij} are elements of $\boldsymbol{W}$. After some manipulation, (8) reduces to the simple form

$$(1+y^2)^{9/2}(\lambda-1)^2(\lambda+1)^2 = 0. \tag{9}$$

This result is somewhat surprising. Generally, in solving an eigenvalue equation of the type of (8), we expect the eigenvalue λ to be a function of y. However, this is not the case here, and we find that solutions of $\lambda = \pm 1$ exist for *all* y. Thus, the generalised weak-graph transformation (4) is always self-dual. The location of the self-dual point will, of course, be y dependent.

The expression (9) is further revealing. It indicates that the determinant in (8) can be diagonalised by a similarity transformation into a form having diagonal elements $\lambda-1$, $\lambda-1$, $\lambda+1$, $\lambda+1$. This means that, for both $\lambda = 1$ and $\lambda = -1$, only 2 of the 4 linear equations in (7) are independent. Therefore, we can eliminate y using any two equations in (7) to obtain the self-dual manifold contaning all fixed (self-dual) points.

It is most convenient to use the first and the last equations in (7). Solving b and d from these two equations, we obtain after some algebra

$$\frac{b}{d} = \frac{a - c - \lambda c\sqrt{1+y^2}}{a + 3c - \lambda a\sqrt{1+y^2}} \qquad b + d = \frac{(a+c)y}{1+\lambda\sqrt{1+y^2}} \qquad \lambda = \pm 1 \tag{10}$$

leading to the relations

$$\begin{aligned} y &= \frac{(b+d)(2ab+3bc-ad)}{(a+c)(ab-cd)} \\ \lambda\sqrt{1+y^2} &= \frac{ab+3bc-ad+cd}{ab-cd} \qquad \lambda = \pm 1. \end{aligned} \tag{11}$$

Substituting the first expression in (11) into the second and squaring both sides, we obtain the self-dual manifold

$$(2ab+3bc-ad)f(a, b, c, d)=0 \qquad \lambda=\pm 1 \tag{12}$$

where $f(a, b, c, d)$ has been given in (3). The vanishing of the first factor in (12) is equivalent to setting $y=0$, for which (4) is an identity transformation†. Therefore, the non-trivial self-dual manifold is precisely (2), obtained previously from a consideration of the Ising equivalence.

Consider next the case of $q=4$, a 16-vertex model whose vertex configurations and weights are shown in figure 2. This 16-vertex model has been considered previously in an Ising subspace (Wu 1972, 1974b). Now, the generalised weak-graph transformation is given by (4) with

$$\tilde{A}=\begin{pmatrix}\tilde{a}\\ \tilde{b}\\ \tilde{c}\\ \tilde{d}\\ \tilde{e}\end{pmatrix} \qquad A=\begin{pmatrix}a\\ b\\ c\\ d\\ e\end{pmatrix}$$
$$W=\frac{1}{(1+y^2)^2}\begin{pmatrix}1 & 4y & 6y^2 & 4y^3 & y^4\\ y & 3y^2-1 & 3y^2-3y & y^4-3y^2 & -y^3\\ y^2 & 2y^3-2y & y^4-4y^2+1 & 2y-2y^3 & y^2\\ y^3 & y^4-3y^2 & 3y-3y^3 & 3y^2-1 & -y\\ y^4 & -4y^3 & 6y^2 & -4y & 1\end{pmatrix}. \tag{13}$$

Using (13), the characteristic equation (8) reduces to

$$-(1+y^2)^8(\lambda-1)^3(\lambda+1)^2=0 \tag{14}$$

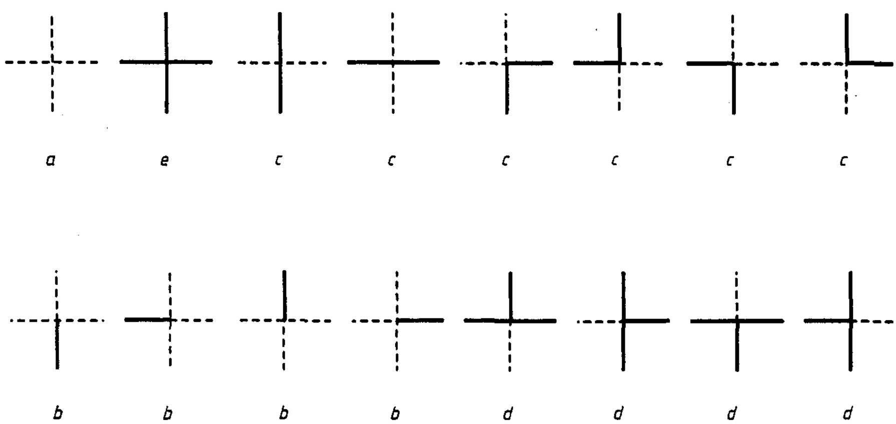

Figure 2. Vertex configurations and weights for the symmetric 16-vertex model.

† The partition function Z is invariant under the negations of b and d (Wu 1974b).

again yielding the result that the generalised weak-graph transformation (4) is self-dual for *all* y, and that (7) yields solution only for $\lambda = \pm 1$.

For $\lambda = 1$, (14) indicates that only two of the five equations in (7) are independent and, consequently, the self-dual manifold is obtained by eliminating y using any two of the five equations. It is again most convenient to use the first and the last equations in (7). By adding and subtracting these two equations, we obtain, respectively,

$$\begin{aligned} a+e &= 2[(d-b)(y^2-1)+3cy]/y \\ a-e &= 2(b+d)/y. \end{aligned} \tag{15}$$

Eliminating y from (15), we obtain the self-dual manifold

$$a^2d - be^2 - 3(a-e)(b+d)c + (b-d)[ae+2(b+d)^2] = 0. \tag{16}$$

It can be shown (Wu and Wu 1988b) that, as in the case of $q=3$ (Wu 1974b), (16) coincides with the critical locus in the ferromagnetic Ising subspace of the vertex model. The present result establishes (16) as the self-dual locus for the whole parameter space.

For $\lambda = -1$, (14) tells us that three of the five equations in (7) are independent. Using any three equations from (7) to eliminate y, we obtain two hypersurfaces in the parameter space, and the self-dual manifold is their intersection. The difference of the first and the last equations in (7) yields

$$y = (e-a)/2(b+d) \tag{17}$$

and the hypersurfaces are then obtained by substituting (17) into any two equations in (7). In practice, however, it proves convenient to use combinations of the five equations which are factorisable after the substitution. After some algebra, we find the following factorisable expressions for the hypersurfaces:

$$\begin{aligned} &(a+2c+e)[(a-e)^2+4(b+d)^2]=0 \\ &[(a-6c+e)(a-e)^2+24c(b+d)^2+4a(3b^2-4bd-5d^2)+4e(3d^2-4bd-5b^2)] \\ &\qquad \times[(a-e)^2-4(b+d)^2]=0. \end{aligned} \tag{18}$$

Note that, unlike the case of $q=3$ for which the self-dual manifold is the same for $\lambda = \pm 1$, (16) and (18) are distinct.

More generally for general q, it can be shown by following the procedure given in Wu (1974b) that the generalised weak-graph transformation (4) is

$$W_{ij} = (1+y^2)^{-q/2} \sum_{k=0}^{j} \binom{i}{k}\binom{q-i}{j-k}(-1)^k y^{i+j-2k} \qquad i, j = 1, 2, \ldots, q+l. \tag{19}$$

We have further evaluated the characteristic equation (8) using this W_{ij} for $q=2$, 5, 6. The results, together with those of $q=3$, 4 given in the above, can be summarised by the equality

$$\det|W_{ij} - \lambda\delta_{ij}| = (-1)^{q+1}(1+y^2)^{q^2/2}(\lambda+1)^{[(q+1)/2]}(\lambda-1)^{[(q+2)/2]} \tag{20}$$

where $[x]$ is the integral part of x, e.g., $[4]=4$, $[\frac{5}{2}]=2$. We conjecture that (20) holds for arbitrary q. It follows from (8) and (20) that the generalised weak-graph transformation (4) is always self-dual.

For $q = 2n =$ even, which is the case in practice for $q > 3$, there are n independent equations in (7) for $\lambda = 1$ and $n+1$ independent equations for $\lambda = -1$. The self-dual manifold will then be the intersection of $n-1$ and n hypersurfaces for $\lambda = 1$ and -1, respectively.

In summary, we have considered the generalised weak-graph transformation for a general vertex model in any dimension. We established that the generalised weak-graph transformation is always self-dual, and obtained the self-dual manifold for $q = 3$, 4. It should be pointed out that this self-dual property is *intrinsic*, since its validity depends only on the fact that there is a uniform coordination number, q, throughout the lattice (thus applying to random lattices with uniform q as well). Consequently, one does not expect to deduce from these considerations physical properties, such as the exact critical temperature of the zero-field Ising model, which are lattice dependent.

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